

# Darboux Transformations for Supersymmetric Korteweg - de Vries Equations

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**ABSTRACT** We consider the Darboux type transformations for the spectral problems of supersymmetric KdV systems. The supersymmetric analogies of Darboux and Darboux-Levi transformations are established for the spectral problems of Manin-Radul-Mathieu sKdV and Manin-Radul sKdV. Several Bäcklund transformations are derived for the MRM sKdV and MR sKdV systems.

# 1 Introduction

Soliton systems, such as KdV equation, Sine-Gordon equation and Nonlinear Schrödinger equation, have been found to enjoy many remarkable properties, for example, they may be solvable via Inverse Scattering Transformation, have Bi-Hamiltonian structure, have infinite number of conservation laws(symmetries) and Bäcklund transformations(BT), etc.. Among many approaches to construct BT for a given system, so-called Classical Darboux Transformation(CDT) is the most direct and yet elementary one. In words, CDT establishes a relationship between solutions of a couple of one dimensional Schrödinger equations or recover some covariance of it[14].

CDT is first used by Wadati, Sanuki and Konno[20] to obtain soliton solutions for the KdV equation. We also notice that the interesting work of Levi, Ragnisco and Sym[10], which shows that CDT is equivalent to the important dressing transformations.

Apart from DT, a new version of Darboux transformation exists. Indeed, Levi[9] introduced a transformation, now known as Darboux-Levi's Transformation(DLT)[1]. Most recent works along this line include DT for KP hierarchy[19][2] and DT in higher dimensional case[19][6]. We refer to Matveev and Salle's book[15] for more examples and applications of DT(see also the review paper in ref.[6]).

The present paper is intended to generalize DT to the context of supersymmetric case. The super KdV equations are introduced recently. We mention the Kupershmidt's version[8] and Manin-Radul's version[11]. While the former is just a fermionic extension of the KdV system, the latter is a genuine supersymmetric KdV(sKdV)[12]. Here, we are only concerned with the supersymmetric case. It is interesting to notice that various structures of sKdV systems are exploited. We just cite the Bi-Hamiltonian theory[18][4][7], the relationship between the second Poisson brackets and  $N = 1$  superconformal algebra[13], Bilinear form[16] and sKdV can be embedded in the supersymmetric Yang-Mills[5]. We will show that CDT and DLT have their analogies in supersymmetric context. We will see that DT and DLT here is less general than the ordinary case due to the pressure of non-commutativity, but they do have interesting consequence: we are able to derive BT for sKdV equations.

The paper is arranged as follows. Next section includes two theorems on DT for MRM sKdV and MR sKdV. The following section is intended to the super version of DLT. In section 4, we calculate some exact solutions for both MRM sKdV equation and MR sKdV equation by means of the Darboux transformations we established in section 2 and last section is a brief summary and some comments.

*Connection:* we follow the usual convention: denoting the fermionic(odd) field variables by Latin letters and bosonic(even) field variables by Greek letters. Also, we always use Greek letters to denote the wave functions whose parities will be stated clearly if necessary.

# 2 Darboux Transformations

Let us start with the Manin-Radul-Mathieu sKdV system[11,12]:

$$\alpha_t = \frac{1}{4}(\alpha_{xxx} + 3(\alpha D\alpha)_x). \quad (1)$$

Where  $D = \theta\partial + \partial_\theta$ . The Lax operator for (1) is:

$$\mathbf{L}_{mrm} = \partial^2 + \alpha D, \quad (2)$$

it is easy to see that the equation(1) is equivalent to:

$$\mathbf{L}_{mrm_t} = [\mathbf{P}_{mrm}, \mathbf{L}_{mrm}], \quad (3)$$

where  $\mathbf{P}_{mrm} = \partial^3 + \frac{3}{2}\alpha D^3 + \frac{3}{4}\alpha_x D$ .

For the system(1), we have:

*Proposition 2.1.*

If  $\phi$  satisfies the equation:

$$\phi_{xx} + \alpha(D\phi) = \lambda\phi, \quad (4)$$

and  $\Lambda$  is a solution of the equation(4) with zero energy. Taking

$$\hat{\phi} = (\partial - \frac{\Lambda_x}{D\Lambda})\phi, \quad (5)$$

then  $\hat{\phi}$  satisfies:

$$\hat{\phi}_{xx} + \hat{\alpha}(D\hat{\phi}) = \lambda\hat{\phi}, \quad (6)$$

with

$$\hat{\alpha} - \alpha = 2(\frac{\Lambda_x}{D\Lambda})_x. \quad (7)$$

Proof. A straightforward calculation.

*Remarks:*

(i). Here, that  $\Lambda$  is a solution of the equation (4) with zero energy is sufficient but not necessary. In fact, if we take  $\Lambda$  as a solution of the equation  $\phi_{xx} + \alpha(D\phi) = \lambda_0\phi$ ,  $\lambda_0 \neq 0$ , then this requires  $\Lambda\Lambda_x = 0$ .

(ii). The proposition makes sense only if  $\Lambda$  is an odd(fermionic) function.

Next, we study application of the Proposition 2.1. Actually, we are able to derive a Bäcklund transformation for MRM sKdV. The equation (7) gives us:

$$\delta = \frac{1}{2}\partial^{-1}(\hat{\alpha} - \alpha) + \eta, \quad (8)$$

where we used the notation  $\delta = \frac{\Lambda_x}{D\Lambda}$  and  $\eta$  is an odd constant.

Also, using the equation  $\Lambda_{xx} + \alpha(D\Lambda) = 0$ , we find:

$$\delta_x + \delta(D\delta) + \alpha = 0, \quad (9)$$

and substitution of (8) into (4) leads to:

$$\hat{\alpha} + \alpha + \frac{1}{2}\alpha(\partial^{-1}(\hat{\alpha} - \alpha) + 2\eta)D^{-1}(\hat{\alpha} - \alpha) = 0. \quad (10)$$

We notice that by redefining the variable, we may put the BT(10) in the differential form:

$$\hat{\beta}_x + \beta_x + \frac{1}{2}\beta_x[\hat{\beta} - \beta + 2\eta][D\hat{\beta} - D\beta] = 0, \text{ with } \alpha = \beta_x \text{ and } \hat{\alpha} = \hat{b}i_x.$$

After the presentation of the Darboux transformation for MRM system(1), we turn to the analogous results for MR's sKdV equation:

$$\begin{aligned}\alpha_t &= \frac{1}{4}(\alpha_{xxx} + 3(\alpha D\alpha)_x + 6(\alpha u)_x), \\ u_t &= \frac{1}{4}(u_{xxx} + 6uu_x + 3\alpha_x(Du) + 3\alpha(Du_x)).\end{aligned}\tag{11}$$

The corresponding Lax operator is given by:

$$\mathbf{L}_{mr} = \partial^2 + \alpha D + u, \tag{12}$$

and the Lax representation is:

$$\mathbf{L}_{mr_t} = [\mathbf{P}_{mr}, \mathbf{L}_{mr}], \tag{13}$$

where  $\mathbf{P}_{mr} = \partial^3 + \frac{3}{2}\alpha D^3 + \frac{3}{2}u\partial + \frac{3}{4}\alpha_x D + \frac{3}{4}u_x$ .

As above, we have:

*Proposition 2.2.*

Let  $(\partial^2 + \alpha D + u)\psi = \lambda\psi$  and  $\Lambda$  is a particular solution of it with  $\lambda = \lambda_0$ . Taking

$$\hat{\psi} = (D + \delta)\psi, \quad \delta = -\frac{D\Lambda}{\Lambda}, \tag{14}$$

then the following equations are satisfied:

$$\hat{\psi}_{xx} + \hat{\alpha}(D\hat{\psi}) + \hat{u}\hat{\psi} = \lambda\hat{\psi}, \tag{15}$$

where

$$\hat{\alpha} + \alpha = -2\delta_x, \quad \hat{u} - u = (D\alpha) + 2\delta\alpha + 2\delta\delta_x, \tag{16}$$

Proof: A direct calculation.

*Remarks:*

- (i).  $\Lambda$  is a bosonic(even) function.
- (ii). Since the operator  $(D + \delta)$  is a fermionic (odd) operator, the wave functions  $\psi$  and  $\hat{\psi}$  have different parities.

A Bäcklund transformation may be obtained for MR's sKdV(11) by means of the proposition 2.2. Let us derive it next.

It is easy to see that the  $\delta$  can be represented as:

$$\delta = -\frac{1}{2}\partial^{-1}(\hat{\alpha} + \alpha) + \eta, \tag{17}$$

where  $\eta$  is an integration constant which is fermionic. Using relation  $D\Lambda = -\delta\Lambda$  and the equation  $\Lambda_{xx} + \alpha(D\Lambda) + u\Lambda = \lambda_0\Lambda$ , we see that  $\delta$  satisfies the following equation:

$$-(D\delta_x) + (D\delta)^2 - \alpha\delta + u = \lambda_0, \tag{18}$$

now, substituting the expression(17) into the equation(17) and the second equation of (16) yields:

$$\begin{aligned} \frac{1}{2}((D\hat{\alpha} + (D\alpha)) + \frac{1}{4}[D^{-1}(\hat{\alpha} + \alpha)]^2 + \frac{1}{2}\alpha\partial^{-1}(\hat{\alpha} + \alpha) - \alpha\eta + u = \lambda_0, \\ u - \hat{u} = (D\alpha) + [-\partial^{-1}(\hat{\alpha} + \alpha) + 2\eta]\alpha - [-\frac{1}{2}\partial^{-1}(\hat{\alpha} + \alpha) + \eta](\hat{\alpha} + \alpha), \end{aligned} \quad (19)$$

which is a Bäcklund transformation for MR's sKdV(11).

As before, we may introduce a new variable and rewrite the above BT in local form.

### 3 New Darboux Transformations

This section is intended to construction of new types of Darboux transformations for both MRM sKdV and MR sKdV. That is, we will show that Levi's generalization of Darboux transformation also has its analogies in the super case.

Indeed, for the MRM sKdV case, we obtain:

*Proposition 3.1*

If  $\phi$  is a solution of the equation  $\phi_{xx} + \alpha(D\phi) = \lambda\phi$  and  $\theta$  is a particular solution of this equation with zero energy. Letting

$$\hat{\phi} = \frac{D^{-1}\Omega}{\theta}, \quad \Omega = (D\phi)\theta - (D\theta)\phi, \quad (20)$$

then,  $\hat{\phi}$  is the solution of the equation:

$$\hat{\phi}_{xx} + \hat{\alpha}\hat{\phi} = \lambda\hat{\phi}, \quad (21)$$

with:

$$\hat{\alpha} = \alpha - 4\left(\frac{D\theta}{\theta}\right)_x. \quad (22)$$

Proof. The calculation involved here is rather tedious. Since the idea employed is similar to the one used next Proposition, we omit the proof here.

*Remarks.*

- (i). The particular solution  $\theta$  must be a bosonic function.
- (ii). This time, the wave function  $\phi$  and  $\hat{\phi}$  have the same parity.
- (iii). To be accurate, we must choose the special solution  $\theta$  such that the quantity:  
 $I = \lambda(\hat{\phi} - \phi)\theta - 2\theta_x\phi_x + 2\theta^{-1}\theta_x^2\phi - 2(D\phi_x)(D\theta) + 2(D\phi)(D\theta_x) + 4\theta^{-1}(D\theta_x)\phi(D\theta).$

It is easy to see that the BT in this case reads

$$(D\hat{\alpha}) - (D\alpha) + \frac{1}{4}[D^{-1}(\hat{\alpha} - \alpha)] + \alpha[\partial^{-1}(\hat{\alpha} - \alpha) + 4\eta] = 0. \quad (23)$$

As for the MR sKdV system(11), we get:

*Proposition 3.2*

Let  $\psi$  and  $\theta$  be solutions of the equation  $(\partial^2 + \alpha D + u)\psi = \lambda\psi$  with the spectral parameters  $\lambda$  and  $\mu$  respectively. Taking

$$\hat{\psi} = \frac{\theta}{\Theta}(D^{-1}(\frac{\Theta\Omega}{\theta^2})) + k\theta\Theta^{-1}, \quad (24)$$

with  $\Theta$  satisfies the following linear equation:

$$\Theta_{xx} - 2\frac{\theta_x}{\theta}\Theta_x + (\alpha - 2\frac{D\theta_x}{\theta} + 2\frac{(D\theta)\theta_x}{\theta^2})(D\Theta) = 0, \quad (25)$$

Then,  $\hat{\psi}$  is the solution of the equation  $(\partial^2 + \hat{\alpha}D + \hat{u})\psi = \lambda\psi$  but with:

$$\hat{\alpha} = \alpha - 2\Gamma_x, \quad \hat{u} = u - 4(D\gamma)(D\Gamma) - 2(D\Gamma)^2 + 2\gamma\Gamma_x + 2\Gamma\gamma_x + 2\Gamma\Gamma_x, \quad (26)$$

where  $\gamma = -\frac{D\theta}{\theta}$ , and  $\Gamma = \frac{D\Theta}{\Theta}$ .

Proof. Let us calculate the quantity:

$$\hat{\psi}_{xx} + \hat{\alpha}(D\hat{\psi}) + \hat{u}\hat{\psi} = (\mu\theta\Theta^{-1})D^{-1}(\Theta\theta^{-2}\Omega) + k\mu\theta\Theta^{-1} + I, \quad (27)$$

where

$$\begin{aligned} I = & 2(\theta\Theta^{-1})_x D(\Theta\theta^{-2}\Omega) + \theta\Theta^{-1} D\partial(\Theta\theta^{-2}\Omega) + \alpha\theta^{-1}\Omega \\ & - 2\Theta^{-1}\theta^{-1}(D\Theta_x)\Omega + 2\Theta^{-2}(D\Theta)\Theta_x\theta^{-1}\Omega, \end{aligned} \quad (28)$$

A long calculation shows that

$$\begin{aligned} I = & -\Theta^{-1}\Theta_x\psi_x + \theta^{-1}\Theta^{-1}\Theta_x\theta_x\psi - \Theta^{-1}(D\Theta_x)(D\psi) + \Theta^{-1}(D\Theta)(D\psi_x) + \\ & \theta^{-1}\Theta^{-1}(D\Theta)(D\psi)\theta_x - \theta^{-1}\Theta^{-1}(D\Theta)\psi_x(D\theta) - \theta^{-1}\Theta^{-1}\psi(D\Theta)(D\theta)_x + \\ & (\psi_{xx} + \alpha(D\psi)) - \theta^{-1}\theta_{xx}\psi - \alpha\theta^{-1}(D\theta)\psi + \theta^{-1}\Theta^{-1}(D\Theta)_x\psi(D\theta). \end{aligned} \quad (29)$$

Thus,

$$\hat{\psi}_{xx} + \hat{\alpha}(D\hat{\psi}) + \hat{u}\hat{\psi} = \lambda\hat{\psi} + \theta\Theta^{-1}[-\lambda\theta^{-1}\Theta\hat{\psi} + \mu D^{-1}(\Theta\theta^{-2}\Omega) + k\mu + \theta^{-1}\Theta I], \quad (30)$$

Now we have to prove that the quantity in the square bracket is a constant. We denote it as  $\hat{I}$ . Then tedious calculation shows:

$$D\hat{I} = 0, \quad (31)$$

thus,  $\hat{I}$  is a constant. This constant can be made as zero by suitable choice of integration constant and suitable boundary conditions.

*Remarks.*

- (i). The wave functions  $\psi$  and  $\hat{\psi}$  have the same parity.
- (ii). A Bäcklund transformation may be derived for the MR sKdV system. However, it is in a very complicated form.

## 4 Exact Solutions of Supersymmetric KdV Systems

Among many applications of Darboux transformations, the most important and direct one is to construct exact solutions for the related nonlinear equations[14]. In the following, we will show that some interesting solutions of sKdV systems may be obtained by means of the propositions 2.1 and 2.2.

In order to use the proposition 2.1 and proposition 2.2 for the sKdV systems, we have to verify that the time evolution of wave functions are covariant under the Darboux transformations, i.e., the equations  $\phi_t = \mathbf{P}_{mr}\phi$  and  $\psi_t = \mathbf{P}_{mrm}\psi$  are covariant under the Darboux transformations given in proposition 2.1 and proposition 2.2 respectively. This indeed can be proved by a long but straightforward calculation.

Let us first consider the MRM sKdV case. The simplest solution of the equation ( ) is the trivial one:  $\alpha = 0$ . With this solution, we find that

$$\Lambda = \theta(ax + b), \quad (32)$$

where a and b are arbitrary(bosonic) constants. Thus, the solution in this case is:

$$\alpha = 2\left(\frac{\theta a}{ax + b}\right)_x, \quad (33)$$

this is a stationary solution of MRM sKdV equation(1).

For the MR sKdV system(11), we also start with the trivial solution  $\alpha = u = 0$ . Then, choosing  $\Lambda$  in the form:

$$\Lambda = (1 + \theta\xi)\cosh(kx + k^3t), \quad (34)$$

where  $\xi$  is a fermionic constant and  $\lambda = k^2$ . The solution of ( ) in this case reads:

$$\alpha = 2\frac{k\theta}{(1 + \theta\xi)\cosh^2(kx + k^3t)}, \quad u = 2\frac{k\xi\theta}{(1 + \theta\xi)^2\cosh^2(kx + k^3t)}. \quad (35)$$

Interestingly if we let  $\theta = \xi$ , we obtain a solution for MRM sKdV system(1) from (35):

$$\alpha = 2\frac{k\theta}{\cosh^2(kx + k^3t)}. \quad (36)$$

To end this section, we remark that many other solutions of sKdV systems may be obtained in this way.

## 5 Conclusion

In this paper, we constructed the supersymmetric generalizations of Darboux transformations. Our main results are neatly presented by four propositions given above. By studying the applications of these results, the Bäcklund transformations and some exact solutions of supersymmetric KdV systems are obtained. Thus we that the supersymmetric versions of Darboux transformations are indeed important.

This is just the starting point of such study and there are many questions to be answered. For instance, it would be interesting to consider the supersymmetric analogies of the Crum's

transformations. Also, it is important to pursue the similar results for other supersymmetric integrable models such as supersymmetric Sine-Gordon, supersymmetric Nonlinear Schrödinger equation, and supersymmetric KP hierarchy, etc.. The results along this line may be presented elsewhere.

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